

• In this Chapter, we shall describe ~~the~~ about the expansion of  $f(x)$  in ascending powers of  $x$  by means of Taylor's Theorem or Maclaurin's Theorem. Taylor's series occupies a fundamental position in any scheme of expansion.

• Taylor's Theorem: - Let  $f(x)$  be a function of  $x$ . If the function  $f(x+h)$  can be expanded in a convergent series of positive integral powers of  $h$ , Then

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$\dots + \frac{h^n}{n!} f^n(x) + \dots \text{ to } \infty.$$

Proof: - It is given that  $f(x+h)$  can be expanded in a series of +ve integral powers of  $h$ .

$$\therefore f(x+h) = A_0 + A_1 h + \frac{A_2 h^2}{2!} + A_3 \frac{h^3}{3!} + \dots + A_n \frac{h^n}{n!} + \dots \quad (1)$$

Where  $A_0, A_1, A_2, A_3, \dots, A_n, \dots$  are functions of  $x$  only i.e. independent of  $h$  and to be determined. We shall assume that the series is Cgt (Convergent) and hence all the derivatives of  $f(x)$  exist.

Now differentiating (1) w.r. to  $h$  we get

$$f'(x+h) = 0 + A_1 + A_2 \cdot \frac{2h}{2!} + A_3 \frac{3h^2}{3!} + \dots + A_n \cdot \frac{nh^{n-1}}{n!} + \dots$$

$$= 0 + A_1 + A_2 h + A_3 \frac{h^2}{2!} + \dots + A_n \frac{h^{n-1}}{(n-1)!} + \dots \quad (2)$$

Again differentiating (2) w.r. to  $h$ , we get

$$f''(x+h) = A_2 + A_3 h + A_4 \frac{h^2}{2!} + \dots \dots \dots (3)$$

Similarly, we can find

$$f'''(x+h) = A_3 + A_4 h + \dots \dots \dots (4)$$

Putting  $h=0$  in (1), (2), (3), (4),  $\dots \dots \dots$ , we get

$$A_0 = f(x), A_1 = f'(x), A_2 = f''(x), \dots \dots \dots$$

Now substituting these values of  $A_0, A_1, A_2, A_3, \dots \dots \dots$  in (1), we get

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots \dots \dots$$
$$\dots \dots + \frac{h^n}{n!} f^n(x) + \dots \dots \dots \text{to } \infty.$$

### Maclaurin's Theorem

• **Statement:** - Let  $f(x)$  be a function of  $x$ . If the function  $f(x)$  can be expanded in a Cgt series of +ve integral powers of  $h$ , then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \dots \dots \text{to } \infty$$

**Proof:** - By Taylor's series, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots \dots \dots$$

Putting  $x=0$  in the above series, we have

$$f(h) = f(0) + h f'(0) + \frac{h^2}{2!} f''(0) + \frac{h^3}{3!} f'''(0) + \dots \dots \dots$$

Now, writing  $h=x$  in above series, we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \dots \dots \text{to } \infty$$

Note: (i) The remainder after  $n$  terms of the series

is given by

$$R_n = \frac{h^n}{n!} f^n(x + \theta h)$$

so that for the convergency of the series  $R_n \rightarrow 0$ .

$R_n$  is called Taylor's remainder after  $n$  terms.

(ii) If  $R_n = \frac{h^n}{n!} f^n(\theta x)$ , where  $0 < \theta < 1$

The above expansion is called the expansion ~~of~~ around  $\theta x = 0$

Q. (1) Use Taylor's Theorem to expand  $e^{x+h}$  in powers of  $h$ .

Solution: - Here, we have

$$f(x+h) = e^{x+h} \text{ and } f(x) = e^x$$

$$\therefore f'(x) = e^x = f''(x) = f'''(x) = \dots$$

By Taylor's theorem, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots$$

$$\therefore e^{x+h} = e^x + he^x + \frac{h^2}{2!} e^x + \dots$$

$$\text{i.e. } e^{x+h} = e^x \left[ 1 + h + \frac{h^2}{2!} + \dots \right]$$

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